10 Sheaf cohomology

Notation: Let $(X, \mathcal{O})$ be a ringed space over a ring R. We denote the category of $\mathcal{O}$-modules by $\mathcal{O}$-Mod and the category of modules over R by R-mod.

The aim is to measure the lack of exactness of the global section functor

$$\Gamma(X,-) : \mathcal{O}\text{-Mod} \to R\text{-mod}, \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) := \mathcal{F}(X)$$

by constructing a δ-functor extending $\Gamma(X,-)$. The general construction of derived functors is outlined in last week's talk. In order to apply the theory presented there, we have to check two properties:

Proposition 10.1. *The functor* $\Gamma(X,-)$ *is left-exact.*

Proof. This was essentially done in talks in the past, since one can compose $\Gamma(X,-)$ as

$$\mathcal{O}\text{-Mod} \xrightarrow{F} \mathrm{Sh}(X,R) \xrightarrow{\text{incl}} \mathrm{PrSh}(X,R) \xrightarrow{-(X)} R\text{-mod}.$$

F is the forgetful functor, which is exact. The inclusion functor is left-exact by [1], chapter 3, 6.9(i) and $-(X)$ is the presheaf global section functor, which is exact by [1], chapter 3, 6.9(iv). □

Proposition 10.2. *The category* $\mathcal{O}$-Mod *has enough injectives.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}$-module. For every $x \in X$, the stalk $\mathcal{F}_x$ at x is an $\mathcal{O}_x$-module. Since $\mathcal{O}_x$ is a ring, and we have proven, that S-mod has enough injectives for any ring S, we find an embedding $\mathcal{F}_x \hookrightarrow \mathcal{E}_x$, where $\mathcal{E}_x$ is an injective $\mathcal{O}_x$-module.
The module $\mathcal{E}_x$ is a sheaf on $\{x\}$ and yields an $\mathcal{O}$-module $j_{x*}\mathcal{E}_x$ via the inclusion $j_x : \{x\} \to X$, where j_{x*} denotes the direct image functor induced by j_x. One can prove, that it is right adjoint to the functor j_x^*, which is induced by the restriction $\mathcal{F} \mapsto \mathcal{F}|_x$.
We define $\mathcal{E} := \prod_{x \in X} j_{x*}\mathcal{E}_x$ (the product in $\mathcal{O}$-Mod). With the universal property of the product and the adjunction $j_{x*} \vdash j_x^*$ we obtain

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, j_{x*}\mathcal{E}_x) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{G}_x, \mathcal{E}_x) \tag{10.1}$$

for every $\mathcal{O}$-module $\mathcal{G}$. By setting $\mathcal{F} = \mathcal{G}$, we get a monomorphism $\mathcal{F} \hookrightarrow \mathcal{E}$ induced by the stalkwise monomorphisms $\mathcal{F}_x \hookrightarrow \mathcal{E}_x$. To prove, that $\mathcal{E}$ is injective, we note that $\mathcal{E}_x$ is injective for every $x \in X$. Hence, $\mathrm{Hom}_{\mathcal{O}_x}(-, \mathcal{E}_x)$ is exact. Then, by (10.1), $\mathrm{Hom}_{\mathcal{O}}(-, \mathcal{E})$ is exact and $\mathcal{E}$ therefore injective. □

Theorem 10.3. *There is a universal* δ*-functor*

$$(H^n(X,-) : \mathcal{O}\text{-Mod} \to R\text{-mod} \ , \ \delta)_{n \geq 0}$$

over $\Gamma(X,-)$*, i.e.,*

(a) *$H^n(X,-)$ are additive functors for $n \geq 0$, with $H^n(X,-) = \Gamma(X,-)$,*

(b) *for every exact sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*of $\mathcal{O}$-modules, there exists a family of R-module homomorphisms $\delta : H^n(X,\mathcal{F}'') \to H^{n+1}(X,\mathcal{F}')$, $n \geq 0$, called **connecting homomorphisms**,*

satisfying

(1) *for every short exact sequence (SES)*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of $\mathcal{O}$-modules, the sequence

$$\begin{array}{l} 0 \longrightarrow \Gamma(X,\mathcal{F}') \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{F}'') \\ \xrightarrow{\delta} H^1(X,\mathcal{F}') \longrightarrow H^1(X,\mathcal{F}) \longrightarrow H^1(X,\mathcal{F}'') \\ \xrightarrow{\delta} H^2(X,\mathcal{F}') \longrightarrow H^2(X,\mathcal{F}) \longrightarrow H^2(X,\mathcal{F}'') \longrightarrow \dots \end{array}$$

is exact

(2) *for every morphism*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}'' & \longrightarrow & 0 \end{array}$$

of SES of $\mathcal{O}$-modules, the diagram

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\delta} & H^n(X,\mathcal{F}') & \longrightarrow & H^n(X,\mathcal{F}) & \longrightarrow & H^n(X,\mathcal{F}'') & \xrightarrow{\delta} & H^{n+1}(X,\mathcal{F}') & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \xrightarrow{\delta} & H^n(X,\mathcal{G}') & \longrightarrow & H^n(X,\mathcal{G}) & \longrightarrow & H^n(X,\mathcal{G}'') & \xrightarrow{\delta} & H^{n+1}(X,\mathcal{G}') & \longrightarrow & \dots \end{array}$$

commutes.

(3) *The δ-functor is universal in the sense, that for any δ-functor $(F^n,\delta)_{n\geq 0}$ extending $\Gamma(X,-)$, there exists a unique family of morphisms of functors $\varphi^n : H^n(X,-) \to F^n$ compatible with δ, such that $\varphi^0 = \mathrm{id}$.*

*A universal δ-functor extending $\Gamma(X,-)$ is unique up to unique isomorphism. $H^n(X,\mathcal{F})$ is called the n^{th} **cohomology of X with coefficients in $\mathcal{F}$**.*

Proof. Talk 9, $H^n(X,-) := R^n\Gamma(X,-)$. $\square$

Remark.
- *In comparison with the p.o.v. in algebraic topology, the $\mathcal{O}$-module is the "variable" in our case here, whereas the topological space X is the "variable" in algebraic topology. In algebraic topology, the construction of, for example, **singular** cohomology is explicit, in the sense, that one explicitly constructs a complex from X and takes the cohomology of this complex. For X locally contractible, one can prove $H^n(X, R_X) \cong H^n_{sing}(X, R)$, where R_X is the locally constant sheaf.*
- *For easier computation, one might choose a greater class of objects to resolve a given $\mathcal{O}$-module, compared to injective resolutions. Obviously, the aim is to compute the same thing! There are some options, e.g., **acyclic resolutions** and some, that particularly work only in $\mathcal{O}$-Mod, i.e., by using "flabby" sheaves, which even give canonical resolutions "Godement resolution" ... more will come next year.*

Definition 10.4. *A $\mathcal{O}$-module $\mathcal{I}$ is called **acyclic**[1], iff $H^n(X,\mathcal{I}) = 0$ for all $n > 0$. For an $\mathcal{O}$-module $\mathcal{F}$, an **acyclic resolution** is an exact sequence $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$, such that $\mathcal{I}^n$ is acyclic for every $n \geq 0$. We denote the sequence by $0 \to \mathcal{F} \to \mathcal{I}^\bullet$.*

Example. *Injective $\mathcal{O}$-modules are acyclic by design and injective resolutions of $\mathcal{O}$-modules are acyclic resolutions.*

Theorem 10.5. *Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an acyclic resolution of an $\mathcal{O}$-module $\mathcal{F}$. Then,*

$$H^n(X,\mathcal{F}) \cong H^n(\Gamma(X,\mathcal{I}^\bullet)),$$

canonical, for all $n \geq 0$. The R-module $H^n(\Gamma(X,\mathcal{I}^\bullet))$ denotes the cohomology of a complex, i.e.,

$$H^n(\Gamma(X,\mathcal{I}^\bullet)) := H^n(0 \to \Gamma(X,\mathcal{I}^0) \to \Gamma(X,\mathcal{I}^1) \to \dots) := \frac{\ker(\Gamma(X,\mathcal{I}^n) \to \Gamma(X,\mathcal{I}^{n+1}))}{\operatorname{im}(\Gamma(X,\mathcal{I}^{n-1}) \to \Gamma(X,\mathcal{I}^n))}.$$

Proof. For the case $n = 0$, we apply $\Gamma(X,-)$ to the exact sequence $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1$. This yields the exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{I}^0) \longrightarrow \Gamma(X,\mathcal{I}^1).$$

Therefore we have

$$H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \ker(\Gamma(X,\mathcal{I}^0) \to \Gamma(X,\mathcal{I}^1)) = H^0(\Gamma(X,\mathcal{I}^\bullet)),$$

which shows the statement. For the case $n \geq 1$, we set $\mathcal{I}^{-1} := \mathcal{K}^0 := \mathcal{F}$ and for $p \geq 1$

$$\mathcal{K}^p := \operatorname{coker}(\mathcal{I}^{p-2} \to \mathcal{I}^{p-1}) \cong \operatorname{im}(\mathcal{I}^{p-1} \to \mathcal{I}^p) = \ker(\mathcal{I}^p \to \mathcal{I}^{p+1}),$$

[1] We can define acyclicity more generally for any left-exact functor $F : \mathcal{C} \to \mathcal{D}$, for $\mathcal{C}$ having enough injectives: an object $\mathcal{I}$ is called F-acyclic, iff $R^nF(\mathcal{I}) = 0$ for all $n > 0$.

where the isomorphism is induced by the map $\mathcal{I}^{p-1} \to \mathcal{I}^p$. From the exact sequence $0 \to \mathcal{K}^p \to \mathcal{I}^p \to \mathcal{I}^{p+1}$, we obtain in the same manner as above

$$H^0(X, \mathcal{K}^p) = \ker(H^0(X, \mathcal{I}^p) \to H^0(X, \mathcal{I}^{p+1})) \tag{10.2}$$

for all $p \geq 1$. This yields also $H^0(X, \mathcal{K}^p) \hookrightarrow H^0(X, \mathcal{I}^p)$ and so

$$\operatorname{im}(H^0(X, \mathcal{I}^{n-1}) \to H^0(X, \mathcal{K}^n)) = \operatorname{im}(H^0(X, \mathcal{I}^{n-1}) \to H^0(X, \mathcal{I}^n)). \tag{10.3}$$

The SES

$$0 \longrightarrow \mathcal{K}^p \longrightarrow \mathcal{I}^p \longrightarrow \mathcal{K}^{p+1} \longrightarrow 0$$

yields the long exact sequence of cohomology

$$\ldots \longrightarrow H^n(X, \mathcal{K}^p) \longrightarrow \overbrace{H^n(X, \mathcal{I}^p)}^{=0} \longrightarrow H^n(X, \mathcal{K}^{p+1}) \longrightarrow H^{n+1}(X, \mathcal{K}^p) \longrightarrow \underbrace{H^{n+1}(X, \mathcal{I}^p)}_{=0} \longrightarrow H^{n+1}(X, \mathcal{K}^{p+1}) \longrightarrow \ldots$$

where the cohomology vanishes as marked above because of acyclicity. Hence, $H^n(X, \mathcal{K}^{p+1}) \cong H^{n+1}(X, \mathcal{K}^p)$. Also, from the long exact sequence (for $p = n - 1$)

$$\ldots \longrightarrow H^0(X, \mathcal{I}^{n-1}) \longrightarrow H^0(X, \mathcal{K}^n) \longrightarrow\!\!\!\!\rightarrow H^1(X, \mathcal{K}^{n-1}) \longrightarrow \underbrace{H^1(X, \mathcal{I}^{n-1})}_{=0} \longrightarrow \ldots,$$

we get

$$H^1(X, \mathcal{K}^{n-1}) \cong \frac{H^0(X, \mathcal{K}^n)}{\operatorname{im}(H^0(X, \mathcal{I}^{n-1}) \to H^0(X, \mathcal{K}^n))}$$

with the homomorphism theorem. Inductive by using the isomorphism derived above, and since $\mathcal{F} = \mathcal{K}^0$, we obtain

$$\begin{aligned} H^n(X, \mathcal{F}) = H^n(X, \mathcal{K}^0) &\cong \ldots \cong H^1(X, \mathcal{K}^{n-1}) \\ &\cong \frac{H^0(X, \mathcal{K}^n)}{\operatorname{im}(H^0(X, \mathcal{I}^{n-1}) \to H^0(X, \mathcal{K}^n))} = H^n(\Gamma(X, I^\bullet)) \end{aligned}$$

where we used (10.2) and (10.3). □

Theorem 10.6. *Let $\Phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces over R. Then, the direct image functor $\Phi_* : \mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod}$ is left exact and there is a universal δ-functor $(R^n\Phi_* : \mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod}, \delta)_{n\geq 0}$ over Φ_*. Hence, for every short exact sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of $\mathcal{O}$-modules, there is a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \Phi_*\mathcal{F}' \longrightarrow \Phi_*\mathcal{F} \longrightarrow \Phi_*\mathcal{F}'' \\ \xrightarrow{\delta} R^1\Phi_*\mathcal{F}' \longrightarrow R^1\Phi_*\mathcal{F} \longrightarrow R^1\Phi_*\mathcal{F}'' \\ \xrightarrow{\delta} R^2\Phi_*\mathcal{F}' \longrightarrow R^2\Phi_*\mathcal{F} \longrightarrow R^2\Phi_*\mathcal{F}'' \longrightarrow \dots \end{array}$$

of $\mathcal{O}_Y$-modules.

Proof. Let $\phi : X \to Y$ be the underlying continuous map of Φ and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a SES in $\mathcal{O}_X$-Mod, $V \subseteq Y$ open and $U := \phi^{-1}(V)$. Then,

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}'').$$

is exact in R-mod and since $\Gamma(U, \mathcal{F}) = \Gamma(V, \Phi_*\mathcal{F})$ (analogue for $\mathcal{F}', \mathcal{F}''$) the sequence

$$0 \longrightarrow \Gamma(V, \Phi_*\mathcal{F}') \longrightarrow \Gamma(V, \Phi_*\mathcal{F}) \longrightarrow \Gamma(V, \Phi_*\mathcal{F}'').$$

is also exact in R-mod. Because V is arbitrary, U is arbitrary and therefore

$$0 \longrightarrow \Phi_*\mathcal{F}' \longrightarrow \Phi_*\mathcal{F} \longrightarrow \Phi_*\mathcal{F}''.$$

is exact in $\mathcal{O}_Y$-Mod. Hence, Φ_* is left exact. The existence of the universal δ-functor follows from the talk last week. □

Remark. *This can be seen as a generalization of the cohomology functor we introduced above. We recover $H^n(X, -)$ by setting $Y = \{*\}$ to be the one-point space and $\mathcal{O}_Y(Y) = R$ (since there is only one stalk). For $\Phi : X \to Y$, the constant morphism, we obtain $R^n\Phi_* = H^n(X, -)$, since we can identify the category of $\mathcal{O}_Y$-modules with the category of R-modules.*

Theorem 10.7. *Let Γ' be the composite functor*

$$\mathcal{O}\text{-Mod} \xrightarrow{F} \mathbb{Z}_X\text{-Mod} \xrightarrow{\Gamma} \mathbb{Z}\text{-mod},$$

where F denotes the forgetful functor, $\Gamma := \Gamma(X, -)$ and $\mathbb{Z}_X$ the constant sheaf with values in $\mathbb{Z}$. Then, there is a natural isomorphism of δ-functors

$$R^n\Gamma' \cong (R^n\Gamma) \circ F = H^n(X, F(-)).$$

In other words, we may compute the cohomology of an $\mathcal{O}$-module, as if it was just a sheaf of Abelian groups and also make use of resolutions in this category.

Proof. We may apply theory from last week's talk. To do so, we have to check, that F transforms injectives in $\mathcal{O}$-Mod into acyclic $\mathbb{Z}_X$-modules, see chapter 5, 3.13 and 3.15 in [1]. □

References

[1] B. R. Tennison, *Sheaf Theory*, Cambridge University Press, **1975**.

[2] T. Wedhorn, *Manifolds, Sheaves, and Cohomology*, Springer Spektrum Wiesbaden, **2016**.

[3] C. Dahlhausen, *Algebra and Topology, part II*, Lecture Notes, **2020**.